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Squeezing by tuning the oscillator frequency

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Abstract. In this paper we examine the possibility of generating squeezed states out of coherent states by external changes of the oscillator frequency. Using the evolution operator method developed by Cheng and Fung, we investigate the evolution of a coherent state of a time-dependent harmonic oscillator as well as its squeezing and coherence property. Two specific models of frequency variation are explored; namely, one whose frequency is exponentially decreasing in time and the other with a periodic frequency. Our analyses indicate that the wavefunction of the time-dependent oscillator starts as a coherent state at $t=0$ and evolves as a squeezed state at a later time. It is also shown that squeezing cannot be generated by an adiabatic change of the oscillator frequency because the variances of $\hat{q}$ and $\hat{p}$ turn out to be adiabatic invariants, whereas a sudden change can produce squeezing.

1. Introduction

In the past few years squeezed states of the electromagnetic field have been widely studied, both theoretically and experimentally [1,2]. These are states which have reduced fluctuations in one field quadrature, when compared with coherent states. Squeezed states of light were first studied by theorists interested in their properties as generalised minimum-uncertainty states [3-11]. These properties were discovered independently by several workers using different terminologies and have been described variously as 'pulsating wavepackets' [3], 'new coherent states' [7,8], 'two-photon coherent states' [9] and 'ideal squeezed states' [11]. The first experimental realisation of squeezed light was reported by Slusher and co-workers [12] using four-wave mixing in sodium atoms. They were able to reduce the optical noise below the vacuum fluctuation level by 7-10% using a combination of phase-stable laser excitation and cavity field enhancement. Since then a tremendous amount of effort has been spent on devising feasible experimentally realisable schemes to generate squeezed states. Recently several laboratories have obtained improved experimental evidence of squeezed states produced by various nonlinear processes [13-15].

A number of possible exciting applications of squeezed states have been suggested. One application proposes using squeezed states of light in optical communication systems to give a signal-to-noise ratio better than the quantum limit of coherent light [16,17]. Another suggested application is in the laser interferometric detection of gravitational radiation [11]. The application of squeezed light will also allow us to modify fundamental vacuum field effects such as spontaneous emission and the Lamb shift.

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It has been shown [18] that a single cavity mode of the electromagnetic field behaves like a simple harmonic oscillator of unit mass and is described by the 'position' and 'momentum' operators $\hat{q}$ and $\hat{p}$ related to the conjugate electric and magnetic field operators $\hat{E}$ and $\hat{H}$. Thus, the discussion of production and detection of squeezed states of the oscillator has immediate relevance in the generation and detection of squeezed light. It is the purpose of this paper to examine the possibility of generating squeezed states out of coherent states by external changes of the oscillator frequency. Using the evolution operator method developed by Cheng and Fung [19], we investigate the evolution of a coherent state of a time-dependent harmonic oscillator and discuss its squeezing and coherence property. Two specific models of frequency variation for the oscillator are examined: one whose frequency is exponentially decreasing in time, and the other with a periodic frequency. Implications of the results are discussed as well.

2. Evolution operator method

Consider a particular time-dependent Hamiltonian which comprises SU(1, 1) group generators

$$\hat{H}(t)=a_1(t)\hat{J}_+ + a_2(t)\hat{J}_0 + a_3(t)\hat{J}_- \tag{1}$$

where $\hat{J}_+$, $\hat{J}_0$ and $\hat{J}_-$ form the SU(1, 1) Lie algebra:

$$\begin{aligned}&[\hat{J}_+,\hat{J}_-]=-2\hat{J}_0\\ &[\hat{J}_0,\hat{J}_\pm]=\pm\hat{J}_\pm\end{aligned} \tag{2}$$

and $a_i(t)$ are arbitrary functions of time. The corresponding Schrödinger equation is

$$\hat{H}(t)|\Phi(t)\rangle = \mathrm{i}\hbar\partial|\Phi(t)\rangle/\partial t. \tag{3}$$

As usual, we will define the evolution operator $\hat{U}(t,0)$ such that

$$|\Phi(t)\rangle=\hat{U}(t,0)|\Phi(0)\rangle \tag{4}$$

where $|\Phi(0)\rangle$ is the wavefunction at time $t=0$. Inserting (4) into (3) yields the evolution equation

$$\begin{aligned}&\hat{H}(t)\hat{U}(t,0)=\mathrm{i}\hbar\partial\hat{U}(t,0)/\partial t\\ &\hat{U}(0,0)=1.\end{aligned} \tag{5}$$

Since $\hat{J}_0$, $\hat{J}_+$ and $\hat{J}_-$ form a closed Lie algebra SU(1, 1), the evolution operator can be expressed in the following form:

$$\hat{U}(t,0)=\exp(c_1(t)\hat{J}_+)\exp(c_2(t)\hat{J}_0)\exp(c_2(t)\hat{J}_-) \tag{6}$$

where $c_i(t)$ are to be determined. Then by direct differentation with respect to time, we obtain

$$\begin{aligned}\partial\hat{U}(t,0)/\partial t=[&(\dot{c}_1-c_1\dot{c}_2+c_1^2\exp(-c_2)\dot{c}_3)\hat{J}_+\\ &+(\dot{c}_2-2c_1\exp(-c_2)\dot{c}_3)\hat{J}_0+\exp(-c_2)\dot{c}_3\hat{J}_-]\hat{U}(t,0).\end{aligned} \tag{7}$$

Substituting (1), (6), and (7) into (5), and comparing the two sides, we obtain three ordinary differential equations:

$$\begin{aligned}&\mathrm{i}\hbar(\dot{c}_1-c_1\dot{c}_2+c_1^2\exp(-c_2)\dot{c}_3)=a_1\\ &\mathrm{i}\hbar(\dot{c}_2-2c_1\exp(-c_2)\dot{c}_3)=a_2\\ &\mathrm{i}\hbar\exp(-c_2)\dot{c}_3=a_3\end{aligned} \tag{8}$$

which can be rewritten as

$$\dot{c}_1 = a'_1 + a'_2 c_1 + a'_3 c_1^2 \tag{9}$$

$$\dot{c}_2 = a'_2 + 2a'_3 c_1 \tag{10}$$

$$\dot{c}_3 = a'_3 \exp(c_2) \tag{11}$$

with the initial conditions

$$c_1(0) = c_2(0) = c_3(0) = 0. \tag{12}$$

The a'_j are given by

$$a'_j = a_j / \mathrm{i}\hbar. \tag{13}$$

Equation (9), which is just the Ricatti equation, is the key equation we have to solve first. Once it is solved, the other two equations can be solved readily to give

$$\begin{aligned} c_2 &= \int_0^t \mathrm{d}u (a'_2 + 2a'_3 c_1) \\ c_3 &= \int_0^t \mathrm{d}u a'_3 \exp(c_2). \end{aligned} \tag{14}$$

3. Time-dependent harmonic oscillator

The general expression for the Hamiltonian of a time-dependent harmonic oscillator of unit mass is

$$\hat{H}(t) = \frac{\hat{p}^2 + \omega(t)^2 \hat{q}^2}{2} \tag{15}$$

where $\omega(t)$ is the oscillator frequency and is time dependent. To tackle this quantum problem, we first rewrite the Hamiltonian in the following form:

$$\hat{H}(t) = a_1(t)\hat{J}_+ + a_2(t)\hat{J}_0 + a_3(t)\hat{J}_- \tag{16}$$

where

$$\begin{aligned} \hat{J}_+ &= \mathrm{i}\frac{\hat{q}^2}{2\hbar} \\ J_- &= \mathrm{i}\frac{\hat{p}^2}{2\hbar} \\ \hat{J}_0 &= \mathrm{i}\frac{\hat{p}\hat{q} + \hat{q}\hat{p}}{4\hbar} \end{aligned} \tag{17}$$

and

$$\begin{aligned} a_1(t) &= \hbar\omega(t)^2/\mathrm{i} \\ a_2(t) &= 0 \\ a_3(t) &= \hbar/\mathrm{i}. \end{aligned} \tag{18}$$

Then we can represent the evolution operator for the above Hamiltonian as follows:

$$\hat{U}(t, 0) = \exp(c_1(t)\hat{J}_+) \exp(c_2(t)\hat{J}_0) \exp(c_3(t)\hat{J}_-) \tag{19}$$

with $c_i(t)$ given by

$$c_1(t) = \frac{\partial \log f(t)}{\partial t} \qquad c_1(0) = 0$$

$$c_2(t) = -2 \log \left| \frac{f(t)}{f(0)} \right| \tag{20}$$

$$c_3(t) = -\int_0^t \mathrm{d}u \left(\frac{f(0)}{f(u)} \right)^2$$

in which $f(t)$ satisfies the following differential equation:

$$\ddot{f}(t) + \omega(t)^2 f(t) = 0. \tag{21}$$

Suppose we start with a coherent state at $t = 0$:

$$|\Phi(0)\rangle = |\alpha\rangle \tag{22}$$

that is

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

with

$$\hat{a} = \frac{\omega_0 \hat{q} + \mathrm{i}\hat{p}}{(2\hbar\omega_0)^{1/2}} \tag{23}$$

$$\omega_0 = \omega(t = 0).$$

The wavefunction at any later time will be represented by

$$|\Phi(t)\rangle = \hat{U}(t, 0)|\alpha\rangle. \tag{24}$$

Now we can define a new operator $\hat{A}$ as

$$\hat{A} = \hat{U}(t, 0)\hat{a}\hat{U}^+(t, 0) \tag{25}$$

and it is easy to see that the wavefunction $|\Phi(t)\rangle$ is a coherent state with respect to this new operator

$$\hat{A}|\Phi(t)\rangle = \alpha|\Phi(t)\rangle. \tag{26}$$

Using (19) and (23), it can be shown that the operator $\hat{a}$ is related to the new operator $\hat{A}$ by a Bogoliubov transformation

$$\hat{A} = \eta_1\hat{a} - \eta_2\hat{a}^+ \tag{27}$$

with

$$|\eta_1|^2 - |\eta_2|^2 = 1 \tag{28}$$

where $\eta_1$and η_2 are given by

$$\eta_1 = \frac{\exp(-c_2/2)}{2} \left(1 - c_1c_3 + \exp(c_2) - \frac{\mathrm{i}c_1}{\omega_0} - \mathrm{i}\omega_0 c_3 \right)$$

$$\eta_2 = \frac{\exp(-c_2/2)}{2} \left(1 + c_1c_3 - \exp(c_2) + \frac{\mathrm{i}c_1}{\omega_0} - \mathrm{i}\omega_0 c_3 \right). \tag{29}$$

These results imply that the wavefunction $|\Phi(t)\rangle$ is a squeezed state. So the wavefunction starts as a coherent state at $t=0$ and evolves as a squeezed state at a later time. To see its squeezing property explicitly, we will compute the variances of $\hat{q}$ and $\hat{p}$. Using the evolution operator in (19), it can be shown that the expectation values of these operators with respect to the wavefunction $|\Phi(t)\rangle$ is given by

$$\begin{aligned}\langle\hat{q}\rangle &= \left(\frac{2\hbar}{\omega_0}\right)^{1/2} \mathrm{Re}[\exp(-c_2/2)(1-\mathrm{i}\omega_0 c_3)\alpha^*] \\ \langle\hat{p}\rangle &= -(2\hbar\omega_0)^{1/2}\,\mathrm{Im}[\exp(-c_2/2)(-c_1c_3+\exp(c_2)-\mathrm{i}c_1/\omega_0)\alpha^*].\end{aligned} \tag{30}$$

The corresponding fluctuations in $\hat{q}$ and $\hat{p}$ will then be

$$\begin{aligned}\Delta q &= \left(\frac{\hbar}{2\omega_0}\right)^{1/2} |\exp(-c_2/2)(1-\mathrm{i}\omega_0 c_3)| \\ \Delta p &= \left(\frac{\hbar\omega_0}{2}\right)^{1/2} |\exp(-c_2/2)(-c_1c_3+\exp(c_2)-\mathrm{i}c_1/\omega_0)|.\end{aligned} \tag{31}$$

Immediately we see that

$$\begin{aligned}\Delta q &\sim |\exp(-c_2/2)| \\ \Delta p &\sim |\exp(c_2/2)|.\end{aligned} \tag{32}$$

So we obtain squeezing in the fluctuation of one operator at the expense of an increase in the fluctuation of the other operator. Thus a squeezing property of $|\Phi(t)\rangle$ is apparent here.

Let us now look at the two limiting cases of an adiabatic change and a sudden change of the oscillator frequency.

3.1. *An adiabatic change of frequency*

In the adiabatic limit of frequency change, we may approximate the desired solution of (21) by

$$f(t) \approx C\cos(\omega(t)t) \tag{33}$$

where C is an arbitrary constant. The corresponding $c_i(t)$ are

$$\begin{aligned}c_1(t) &\approx -\omega(t)\tan(\omega(t)t) \\ c_2(t) &\approx -2\log|\cos(\omega(t)t)| \\ c_3(t) &\approx \frac{-1}{\omega(t)}\tan(\omega(t)t).\end{aligned} \tag{34}$$

Then it is easy to see that, in the lowest approximation, Δq and Δp at time t are given by

$$\Delta p = \omega(t)\Delta q = \left(\frac{\hbar\omega(t)}{2}\right)^{1/2}. \tag{35}$$

This implies that the variances of $\hat{q}$ and $\hat{p}$ are adiabatic invariants. Thus, squeezing cannot be generated by adiabatic changes in the oscillator frequency.

3.2. A sudden change of frequency

Suppose that $\omega(t)$ has a sudden jump at time t_0, namely, $\omega(0 \leqslant t \leqslant t_0) = \omega^-$ and $\omega(t > t_0) = \omega^+$. Then, according to the sudden approximation [20], the evolution operator $\hat{U}(t_0^+, t_0^-) = 1$; this implies that $c_i(t_0^+) = c_i(t_0^-)$. Without loss of generality, we will take $t_0 = (2\pi/\omega^-)$ for simplicity. Consequently, it is not difficult to show that, for $0 \leqslant t \leqslant t_0$,

$$c_1(t) = -\omega^- \tan(\omega^- t)$$
$$c_2(t) = -2 \log|\cos(\omega^- t)|$$
$$c_3(t) = \frac{-1}{\omega^-} \tan(\omega^- t)$$

and for $t > t_0$,

$$\begin{aligned} c_1(t) &= -\omega^+ \tan(\omega^+(t - t_0)) \\ c_2(t) &= -2 \log|\cos(\omega^+(t - t_0))| \\ c_3(t) &= \frac{-1}{\omega^+} \tan(\omega^+(t - t_0)). \end{aligned} \tag{36}$$

Hence, Δq and Δp at time t are given by

$$\begin{aligned} \Delta q(0 \leqslant t \leqslant t_0) &= (\omega^+/\omega^-)^{1/2} \Delta q(t > t_0) = (\hbar/2\omega^-)^{1/2} \\ \Delta p(0 \leqslant t \leqslant t_0) &= (\omega^-/\omega^+)^{1/2} \Delta p(t > t_0) = (\hbar\omega^-/2)^{1/2}. \end{aligned} \tag{37}$$

Clearly, squeezing will occur in either Δq or Δp as a result of a sudden change of oscillator frequency.

4. Model Hamiltonian

In this section two specific models of frequency variation for the time-dependent oscillator will be considered.

4.1. Exponentially decreasing frequency ($\omega(t) = \exp(-\varepsilon t/2)$, $0 < \varepsilon \ll 1$)

With this model frequency, (21) becomes

$$\ddot{f}(t) + \exp(-\varepsilon t) f(t) = 0. \tag{38}$$

Introducing the variable

$$s(t) = \frac{2}{\varepsilon} \exp(-\varepsilon t/2) \tag{39}$$

we can reduce (38) to a zeroth-order Bessel differential equation

$$\frac{d^2 f}{ds^2} + \frac{1}{s} \frac{df}{ds} + f = 0 \tag{40}$$

whose general solution is, of course, given by

$$f(s) = k_1 J_0(s) + k_2 Y_0(s) \tag{41}$$

for some arbitrary constants k_1 and k_2. Here, $J_0(s)$ and $Y_0(s)$ are the zeroth-order Bessel functions of the first and second kind, respectively. Thus $c_i(t)$ can be expressed as

$$\begin{aligned} c_1(t) &= \frac{\varepsilon s}{2}\frac{k_1 J_1(s)+k_2 Y_1(s)}{k_1 J_0(s)+k_2 Y_0(s)} \\ c_2(t) &= -2\log\left|\frac{k_1 J_0(s)+k_2 Y_0(s)}{k_1 J_0(2/\varepsilon)+k_2 Y_0(2/\varepsilon)}\right| \\ c_3(t) &= -\int_0^t \mathrm{d}u\left(\frac{k_1 J_0(2/\varepsilon)+k_2 Y_0(2/\varepsilon)}{k_1 J_0(s)+k_2 Y_0(s)}\right)^2. \end{aligned} \tag{42}$$

In order to satisfy the initial condition $c_1(0)=0$, we must require

$$B \equiv \frac{k_2}{k_1} = -\frac{J_1(2/\varepsilon)}{Y_1(2/\varepsilon)}. \tag{43}$$

With the large argument asymptotic expansion of the Bessel functions, it can be shown that

$$B \approx \tan\left(\frac{2}{\varepsilon}-\frac{\pi}{4}\right)$$

and

$$J_0(2/\varepsilon)+BY_0(2/\varepsilon) \approx \left(\frac{\varepsilon}{\pi}\right)^{1/2}\sec\left(\frac{2}{\varepsilon}-\frac{\pi}{4}\right). \tag{44}$$

In the small-t limit, namely, $s(t) \gg 1$, the $c_i(t)$ are given by

$$\begin{aligned} c_1(t) &\approx -\tan(t)\exp(-\varepsilon t/2) \\ c_2(t) &\approx 2\log|\sec(t)| - \tfrac{1}{2}\varepsilon t \\ c_3(t) &\approx -\tan(t)\exp(-\varepsilon t/2). \end{aligned} \tag{45}$$

Then, in the lowest approximation, we obtain

$$\Delta p = \omega(t)\Delta q. \tag{46}$$

This result is, of course, exactly what we expect from an adiabatic change of oscillator frequency. In the large-t limit, i.e. $s(t) \ll 1$, the $c_i(t)$ are given by

$$\begin{aligned} c_1(t) &\sim \frac{1}{t} \\ c_2(t) &\sim -\log\left[\frac{\varepsilon}{\pi}\sin^2\left(\frac{2}{\varepsilon}-\frac{\pi}{4}\right)t^2\right] \\ c_3(t) &\sim \pi\left[\varepsilon\sin^2\left(\frac{2}{\varepsilon}-\frac{\pi}{4}\right)t\right]^{-1}. \end{aligned} \tag{47}$$

With the above results, we can obtain Δq and Δp as follows:

$$\begin{aligned} \Delta q &\sim \left[\frac{\hbar\varepsilon}{2\pi}\sin^2\left(\frac{2}{\varepsilon}-\frac{\pi}{4}\right)\right]^{1/2} t \gg \left(\frac{\hbar}{2}\right)^{1/2} \\ \Delta p &\sim \left[\frac{\hbar\varepsilon}{2\pi}\sin^2\left(\frac{2}{\varepsilon}-\frac{\pi}{4}\right)\right]^{1/2} \ll \left(\frac{\hbar}{2}\right)^{1/2}. \end{aligned} \tag{48}$$

It is apparent that strong squeezing occurs in the fluctuation of $\hat{p}$ in the large-t limit.

All in all, the above analyses explicitly illustrate that, in the small-t limit, when the adiabatic approximation is valid, there is no squeezing; whereas in the large-t limit, when the process is no longer adiabatic, squeezing is generated.

4.2. *Periodic frequency* ($\omega(t)=[\beta-2\gamma\cos(2\varepsilon t)]^{1/2}$, $0<2\gamma<\beta$, $\varepsilon>0$)

Introducing this model frequency into (21) produces

$$\ddot{f}(t)+[\beta-2\gamma\cos(2\varepsilon t)]f(t)=0. \tag{49}$$

Equation (49) can be rewritten in the canonical form of the Mathieu equation:

$$\mathrm{d}^2f/\mathrm{d}\tau^2+[A-2Q\cos(2\tau)]f=0 \tag{50}$$

where

$$\begin{aligned}\tau&=\varepsilon t\\ A&=\beta/\varepsilon^2\\ Q&=\gamma/\varepsilon^2.\end{aligned} \tag{51}$$

Suppose we assume that A is the characteristic number for the second-order cosine-type Mathieu function of the first kind $ce_2(\tau,Q)$. Then the general solution of (50) is given by

$$f(\tau)=k_3ce_2(\tau,Q)+k_4fe_2(\tau,Q) \tag{52}$$

for some arbitrary constants k_3 and k_4. Here, $fe_2(\tau,Q)$ is the second-order Mathieu function of the second kind conjugate to $ce_2(\tau,Q)$. In order to satisfy the initial condition $c_1(0)=0$, we must require $k_4=0$. Thus $c_i(t)$ can be expressed as

$$\begin{aligned}c_1(t)&=\varepsilon\frac{ce_2'(\tau,Q)}{ce_2(\tau,Q)}\\ c_2(t)&=-2\log\left|\frac{ce_2(\tau,Q)}{ce_2(0,Q)}\right|\\ c_3(t)&=-\int_0^t\mathrm{d}u\left(\frac{ce_2(0,Q)}{ce_2(\tau,Q)}\right)^2.\end{aligned} \tag{53}$$

Now, using the above results, we can write Δq and Δp as follows:

$$\begin{aligned}\Delta q&=\left(\frac{\hbar}{2\omega_0}\right)^{1/2}\left|\frac{ce_2(\tau,Q)}{ce_2(0,Q)}\right|\left|1+\mathrm{i}\frac{\omega_0}{\varepsilon}f(\tau)\right|\\ \Delta p&=\left(\frac{\hbar\omega_0}{2}\right)^{1/2}\left|\frac{ce_2(0,Q)}{ce_2(\tau,Q)}\right|\left|1+\frac{ce_2'(\tau,Q)ce_2(\tau,Q)}{(ce_2(0,Q))^2}f(\tau)-\mathrm{i}\frac{\varepsilon}{\omega_0}\frac{ce_2'(\tau,Q)ce_2(\tau,Q)}{(ce_2(0,Q))^2}\right|\end{aligned} \tag{54}$$

where

$$\begin{aligned}f(\tau)&=\int_0^\tau\mathrm{d}\tau'\left(\frac{ce_2(0,Q)}{ce_2(\tau',Q)}\right)^2\\ \omega_0/\varepsilon&=(A-2Q)^{1/2}.\end{aligned} \tag{55}$$

For $Q \leqslant 1$, it can be shown that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{se_2(\tau, Q)}{ce_2(\tau, Q)}\right) \approx \frac{1}{(ce_2(\tau, Q))^2}\left[1 - \frac{Q^2}{32}\left(\cos(4\tau) - \frac{4}{9}\right) + \text{higher-order terms in } Q\right] \tag{56}$$

where $se_2(\tau, Q)$ is the second-order sine-type Mathieu function of the first kind. This implies that $f(\tau)$ can be approximated by

$$f(\tau) \approx \frac{(ce_2(0, Q))^2}{2}\frac{se_2(\tau, Q)}{ce_2(\tau, Q)}. \tag{57}$$

Accordingly, for $Q \leqslant 1$, Δq and Δp are given in closed-form by

$$\begin{aligned} \Delta q &= \left(\frac{\hbar}{2\omega_0}\right)^{1/2}\left|\frac{ce_2(\tau, Q)}{ce_2(0, Q)}\right|\left|1 + \mathrm{i}\frac{(A-2Q)^{1/2}(ce_2(0, Q))^2}{2}\frac{se_2(\tau, Q)}{ce_2(\tau, Q)}\right| \\ \Delta p &= \left(\frac{\hbar\omega_0}{2}\right)^{1/2}\left|\frac{ce_2(0, Q)}{ce_2(\tau, Q)}\right|\left|1 + \frac{ce_2'(\tau, Q)se_2(\tau, Q)}{2} - \mathrm{i}\frac{ce_2'(\tau, Q)ce_2(\tau, Q)}{(A-2Q)^{1/2}(ce_2(0, Q))^2}\right|. \end{aligned} \tag{58}$$

In figures 1 and 2, we plot the time variation of the fluctuations of Δq and Δp, respectively. It should be noted that, since both $ce_2(\tau, Q)$ and $se_2(\tau, Q)$ are periodic functions of τ with period π, Δq and Δp are also periodic in τ with the same period. These diagrams clearly show that there is squeezing in the fluctuation of $\hat{q}$, together with an increase in the fluctuation of $\hat{p}$. Furthermore, we can easily see that here the squeezing in Δq persists at all times, and that the degree of squeezing can be changed by varying the parameter Q.

In summary, the above analyses of the two specific models illustrate explicitly that the time-dependent oscillator starts as a coherent state at $t = 0$ and evolves as a squeezed state at a later time, as well as that an adiabatic change of the oscillator frequency cannot generate squeezing because the variances of $\hat{q}$ and $\hat{p}$ are adiabatic invariants.

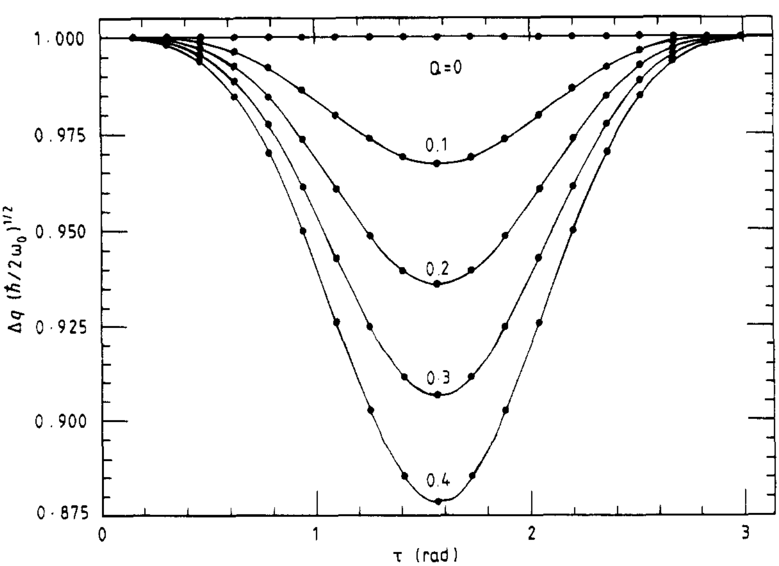

Figure 1. Time variation of the variance in q, Δq.

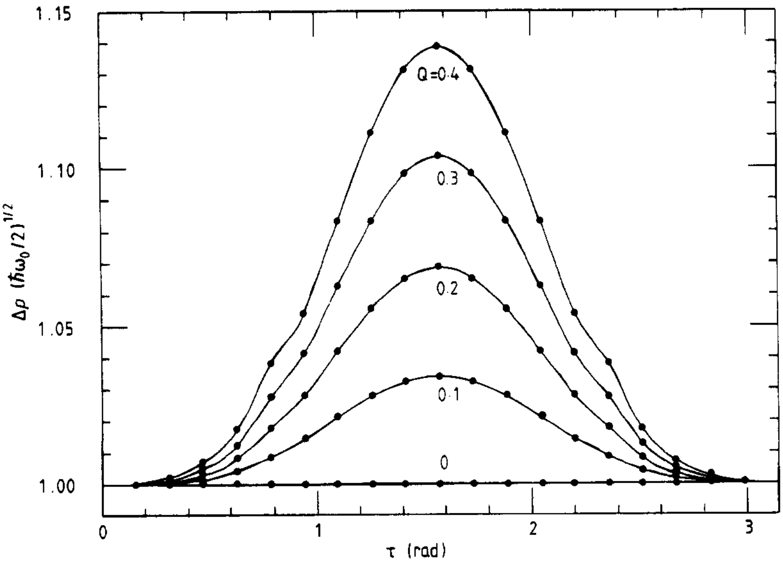

Figure 2. Time variation of the variance in p, Δp.

5. Conclusion

We have investigated the evolution of a coherent state of a time-dependent harmonic oscillator as well as its squeezing and coherence property using the evolution operator method developed by Cheng and Fung. Two specific models of frequency variation were explored; namely, one whose frequency is exponentially decreasing in time and the other with a periodic frequency. Our analyses indicate that the wavefunction of the time-dependent oscillator starts as a coherent state at $t = 0$ and evolves as a squeezed state at a later time. We also observed that squeezing cannot be generated by an adiabatic change of the oscillator frequency because the variances of $\hat{q}$ and $\hat{p}$ turn out to be adiabatic invariants, whereas a sudden change can produce squeezing. Hence, it can be concluded that squeezed states can be generated out of coherent states by external changes of the oscillator frequency.

Note added. Squeezing due to changes in the oscillator frequency has been discussed in the literature. It was shown by Janszky and Yushin [21] that a sudden frequency jump leads to squeezing. Also, the possibility of generating squeezing by continuous change of oscillator frequency has been considered recently by Ma and Rhodes [22]. I should like to thank the referees for bringing these references to my attention.

References

[1] Henry R W and Glotzer S C 1988 *Am. J. Phys.* **56** 318
[2] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709
[3] Takahasi H 1965 *Adv. Commun. Syst.* **1** 227
[4] Robinson D R 1965 *Commun. Math. Phys.* **1** 159
[5] Stoler D 1970 *Phys. Rev.* D **1** 3217
[6] Stoler D 1971 *Phys. Rev.* D **4** 1925
[7] Lu E Y C 1971 *Lett. Nuovo Cim.* **2** 1241
[8] Lu E Y C 1972 *Lett. Nuovo Cim.* **3** 585
[9] Yuen H P 1976 *Phys. Rev.* A **13** 2226

[10] Hollenhorst J N 1979 *Phys. Rev.* D **19** 1669
[11] Caves C M 1981 *Phys. Rev.* D **23** 1693
[12] Slusher R E, Hollberg L W, Yurke B, Mertz J C and Valley J F 1985 *Phys. Rev. Lett.* **55** 2409
[13] Robinson R L 1986 *Science* **233** 280
[14] Shelby R M, Levenson M D, Perlmutter S H, DeVoe R G and Walls D F 1986 *Phys. Rev. Lett.* **57** 691
[15] Wu L A, Kimble H J, Hall J L and Wu H 1986 *Phys. Rev. Lett.* **57** 2520
[16] Yuen H P and Shapiro J H 1978 *IEEE Trans. Inf. Theory* **IT-24** 657
[17] Shapiro J H, Yuen H P and Machado Mata J A 1979 *IEEE Trans. Inf. Theory* **IT-25** 179
[18] Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)
[19] Cheng C M and Fung P C W 1988 *J. Phys. A: Math. Gen.* **21** 4115
[20] Schiff L I 1968 *Quantum Mechanics* (New York: McGraw-Hill)
[21] Janszky J and Yushin Y Y 1986 *Opt. Commun.* **59** 151
[22] Ma X and Rhodes W 1989 *Phys. Rev.* A **39** 1941